An introduction to the Hybrid High-Order method and its applications on Maxwell's equations

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Intro

Hybrid High-Order (HHO) methods are a recent development [Di Pietro, Ern, Lemaire 2014] in the family of Discontinuous Skeletal methods (HDG [Cockburn et al. 2009], WG [Wang et al. 2013], ...)

- Arbitrary order
- Any element shape
- Dimension-independent formulation
- Simple *hp*-refinement

HHO is well established: wide literature (mostly in mechanics) + 2 books. In this talk:

- Intro to HHO on the Poisson equation
- HHO for time-harmonic Maxwell
- A real-world application of HHO in electromagnetics

Model problem

First part of this talk: Poisson equation with homogeneous BCs. Let $\Omega \subset \mathbb{R}^d$ with $d \in \{1, 2, 3\}$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

In weak form, for $f \in L^2(\Omega)$ find $u \in H^1_0(\Omega)$ s.t.

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)} \qquad \forall v \in H^1_0(\Omega)$$

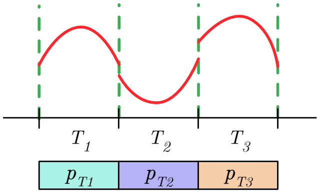
In the following we assume that Ω is partitioned with an appropriate polyhedral mesh $\mathcal{M}(\mathcal{T}, \mathcal{F})$.

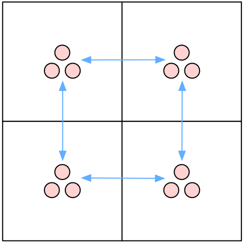
Discontinuous Galerkin recall: discrete space, stencil

Recall that the discrete DG space is made of polynomials of degree k attached to each mesh cell T:

$$V_h := \{v \in L^2(\Omega) \mid \forall T \in \mathcal{M}, v_{|T} \in \mathbb{P}_d^k(T)\}.$$

- Volume term approximating solution locally + coupling via numerical fluxes. A cell T talks with **all** the adjacent cells.
- Global discrete solution is discontinuous.





Number of DOFs grows like $O(\#\mathcal{T} \cdot k^d)$. Can we do better?

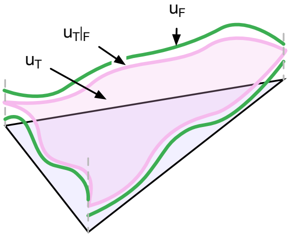
Reconstruction operator - I

Consider the divergence theorem on an element T

$$(\nabla u, \nabla v)_T = (u, \Delta v)_T + \sum_{F_i \in \partial T} (u, \nabla v \cdot \hat{\mathbf{n}})_{F_i}$$

Replace u by different functions on the cell and its faces, and introduce the operator R

$$(\nabla R(u_T, u_{\partial T}), \nabla v)_T := (u_T, \Delta v)_T + \sum_{F_i \in \partial T} (u_{F_i}, \nabla v \cdot \hat{\mathbf{n}})_{F_i}$$
$$= (\nabla u_T, \nabla v)_T + \sum_{F_i \in \partial T} (u_{F_i} - u_T, \nabla v \cdot \hat{\mathbf{n}})_{F_i}$$



- u_T: cell-based function
- u_{F_i}: face-based function
- $u_{\partial T}$: $(u_{F_1}, \ldots, u_{F_n})$

We call R as defined *reconstruction operator*. Note that the operator is completely local.

Reconstruction operator - II

More precisely, let $u_T \in \mathbb{P}_d^k(T)$, $u_{F_i} \in \mathbb{P}_{d-1}^k(F_i)$ and $v \in \mathbb{P}_d^{k+1}(T)$. We define the local HHO space

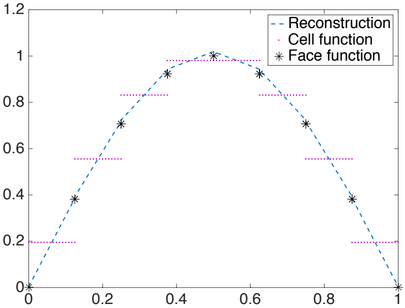
$$U_T^k := \mathbb{P}_d^k(T) \times \left\{ \bigtimes_{F_i \in \partial T} \mathbb{P}_{d-1}^k(F_i) \right\}.$$

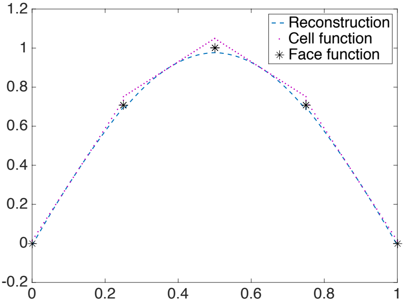
Let $\underline{u}_T := (u_T, u_{\partial T}) \in U_T^k$. The reconstruction $R : U_T^k \to \mathbb{P}_d^{k+1}(T)$ is uniquely defined for all $\underline{u}_T \in U_T^k$ by the equations

$$(\nabla R(\underline{u}_T), \nabla v)_T = (\nabla u_T, \nabla v)_T + \sum_{F \in \partial T} (u_{F_i} - u_T, \nabla v \cdot \hat{\mathbf{n}})_F$$
$$(R(\underline{u}_T), 1)_T = (u_T, 1)_T$$

R enjoys an high-order approximation property: from u_T and $u_{\partial T}$ of degree k we can reconstruct a polynomial of order $k+1$ on the cell.

Reconstruction operator - III





The reconstruction operator is used to mimic the grad-grad term of our model problem

$$a_T(\underline{u}_T, \underline{v}_T) := (\nabla R(\underline{u}_T), \nabla R(\underline{v}_T))_T$$

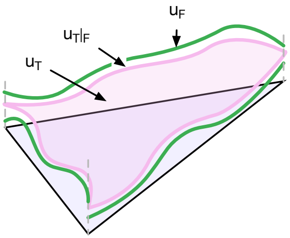
Note that inside this term hides a degree $k+1$ stiffness matrix.

Stabilization - I

There is an issue however: u_T and $u_{\partial T}$ are still unrelated $\implies$ $\nabla R = 0$ does **not** imply $u_T = u_{\partial T} = constant$ $\implies$ a stabilization is needed.

We penalize the difference between u_F and the trace of u_T. First try:

$$z_T(\underline{u}_T, \underline{v}_T) := \sum_{F_i \in \partial T} h_{F_i}^{-1} (u_{F_i} - \pi_{F_i}^k(u_T), v_{F_i} - \pi_{F_i}^k(v_T))_{Fi}$$



- Ask to "glue together" u_F and u_T on each face
- Works, but insufficient to achieve optimal convergence rate

Stabilization - II

There are two alternative ways to achieve optimal convergence rate:

- Use a more complex stabilization

$$s_T(\underline{u}_T, \underline{v}_T) := \sum_{F_i \in \partial T} h_F^{-1}(u_{F_i} - \pi_{F_i}^k P^k(\underline{u}_T), v_{F_i} - \pi_{F_i}^k P^k(\underline{v}_T))_{F_i}$$

$$P^k(\underline{w}_T) := w_T - R(\underline{w}_T) + \pi_T^k(R(\underline{w}_T))$$

  Essentially, it takes into account high-order components

- Take $u_T \in \mathbb{P}_d^{k+1}(T)$ and $u_{F_i} \in \mathbb{P}_{d-1}^k(F_i)$

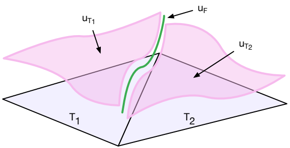
On standard cells, method 2 makes assembly slightly cheaper.

Both alternatives allow the method to reach convergence rate $O(h^{k+2})$ in L^2 norm and $O(h^{k+1})$ in H^1 norm.

Note that again, stabilizations are completely local.

Discrete HHO space, assembly

The global HHO space is obtained collecting cell and face DOFs



$$\underline{U}_h^k := \left\{ \bigtimes_{T\in\mathcal{T}} \mathbb{P}_d^k(T) \right\} \times \left\{ \bigtimes_{F\in\mathcal{F}} \mathbb{P}_{d-1}^k(F) \right\}.$$

Dirichlet BCs imposed strongly as $\mathrm{U}_{h,0}^k := \left\{ \underline{u}_h \in \underline{U}_h^k \mid u_F = 0 \quad \forall F \in \Gamma \right\}$.

Note that the face DOFs are single valued. Let L_T be the standard local-to-global DOF mapping. By standard FEM assembly we compute

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}} a_T(L_T \underline{u}_T, L_T \underline{v}_T) + s_T(L_T \underline{u}_T, L_T \underline{v}_T),$$

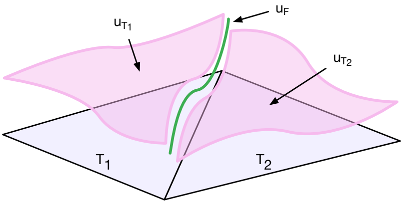
$$l_h(\underline{v}_h) := \sum_{T\in\mathcal{T}} (f, (L_T u_T, 0))_T.$$

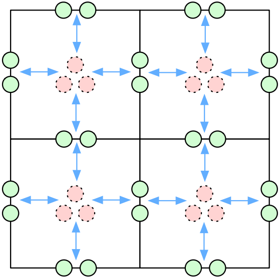
We finally look for $\underline{u}_h \in \mathrm{U}_{h,0}^k$ such that

$$a_h(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h), \qquad \forall \underline{v}_h \in \mathrm{U}_{h,0}^k.$$

HHO stencil

Remember that HHO operators are defined locally on cells: this means that cell unknowns talk only with face unknowns.





- Cell unknowns can be eliminated **locally** during assembly via Schur complement.
- The global problem is posed in terms of face unknowns only
- No. of DOFs grows like $O(\#\mathcal{F} \cdot k^{d-1})$ vs. $O(\#\mathcal{T} \cdot k^{d})$ of DG $\implies$ we expect an improvement over DG on standard elements.

Does HHO pay off?

Poisson equation on $\Omega = [0,1]^3$. Solver: PARDISO, memory in MB.

	HHO(k,k)			SIP-DG($k+1$)		
k	DoFs	Mflops	Mem	DoFs	Mflops	Mem
0	5760	38	39	12288	787	85
1	17280	1006	106	30720	11429	319
2	34560	8723	292	61440	92799	1108
3	57600	40389	719	107520	497245	3215

Tetrahedral mesh, 3072 elements. k=3: HHO is 12.3x more efficient in computation, 4.5x more efficient in memory usage.

	HHO(k, k)			SIP-DG($k+1$)		
k	DoFs	Mflops	Mem	DoFs	Mflops	Mem
0	11520	310	64	16384	6677	168
1	34560	9671	293	40960	104199	765
2	69120	58977	884	81920	845545	2844
3	115200	349664	2412	143360	4592328	8490

Hexahedral mesh, 4096 elements. k=3: HHO is 13.1x more efficient in computation, 3.5x more efficient in memory usage.

HHO on indefinite Maxwell problem

We want to solve the electromagnetic time-harmonic wave equation

$$(\nabla \times \mathbf{e}, \nabla \times \mathbf{v})_\Omega - \omega^2 \mu\epsilon(\mathbf{e}, \mathbf{v})_\Omega = (\mathbf{f}, \mathbf{v})_\Omega.$$

Quantities appearing in the equation:

- ω: angular frequency
- μ, ϵ: piecewise constant material parameters
- $\mathbf{e}, \mathbf{v} \in H_0(curl; \Omega)$: unknown electric field and test function
- $\mathbf{f}$: source

Motivation: curl-curl is difficult for iterative solvers, direct solvers are usually employed $\implies$ being memory-efficient is imperative.

HHO function spaces

Local HHO function space employs vector-valued polynomials:

$$U_T^k := \mathbb{P}_3^k(T)^3 \times \left\{ \bigtimes_{F \in \partial T} \mathbb{P}_2^k(F)^2 \right\}.$$

- Cell-based polynomials have values in $\mathbb{C}^3$
- Face-based polynomials have values in $\mathbb{C}^2$ tangent to the face itself $\Longrightarrow$ reflects tangential continuity of $\mathbf{e}$ at the continuous level

The global discrete problem space is introduced as

$$U_h^k := \left\{ \bigtimes_{T \in \mathcal{T}} \mathbb{P}_3^k(T)^3 \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}} \mathbb{P}_2^k(F)^2 \right\},$$

Dirichlet conditions on $\Gamma \subset \partial\Omega$ are imposed by forcing to zero face DOFs

$$\mathrm{U}_{h,0}^k := \left\{ \underline{\mathrm{u}}_h \in U_h^k \mid \mathrm{u}_F = 0 \quad \forall F \in \Gamma \right\}.$$

HHO operators

In the same spirit of the reconstruction used for Poisson equation, we define the curl reconstruction as

$$(\mathcal{C}(\underline{\mathrm{u}}_T), \mathbf{v})_T := (\mathrm{u}_T, \nabla \times \mathbf{v})_T + \sum_{F \in \partial T} (\mathrm{u}_F, \mathbf{v} \times \hat{\mathbf{n}})_F, \quad \forall \mathbf{v} \in \mathbb{P}_3^k(T)^3$$

Let $\gamma_{t,F}(\mathbf{u}) := \hat{\mathbf{n}} \times (\mathbf{u} \times \hat{\mathbf{n}})$ and $\pi_\gamma^k = \pi_F^k \circ \gamma_{t,F}$. We define the stabilization

$$s_T(\underline{\mathrm{u}}_T, \underline{\mathrm{v}}_T) := \sum_{F \in \partial T} \frac{\omega^2 \mu \epsilon}{h_F} (\mathrm{u}_F - \pi_\gamma^k(\mathrm{u}_T), \mathrm{v}_F - \pi_\gamma^k(\mathrm{v}_T))_F,$$

with the aim of penalizing the difference between the face function and the tangential component of the cell function.

The method is not superconvergent: $O(h^{k+1})$ in L^2 norm and $O(h^k)$ in energy norm. Superconvergence is a work in progress.

Global problem

Local element contributions are given by

$$a_T(\underline{u}_T, \underline{v}_T) := (\mathcal{C}(\underline{u}_T), \mathcal{C}(\underline{v}_T))_T - \omega^2 \mu\epsilon((u_T, 0), (v_T, 0))_T + s_T(\underline{u}_T, \underline{v}_T)$$
$$l_T(\underline{v}_T) := (\mathbf{f}, (v_T, 0))_T$$

Again, static condensation is possible. Global bilinear forms are obtained by adding the local contributions

$$a_h(\underline{u}_h, \underline{v}_h) := \sum_{T\in\mathcal{T}} a_T(L_T\underline{u}_T, L_T\underline{v}_T) \qquad l_h(\underline{v}_h) := \sum_{T\in\mathcal{T}} (\mathbf{f}, (L_T u_T, 0))_T$$

We finally look for $\underline{u}_h \in \mathrm{U}^k_{h,0}$ such that

$$a_h(\underline{u}_h, \underline{v}_h) = l_h(\underline{v}_h) \qquad \forall \underline{v}_h \in \mathrm{U}^k_{h,0}$$

HHO performance

Resonator $[0, 1]^3$, tetrahedral mesh, 3072 elements.

Degree	HHO(k,k) Memory	HHO(k,k) Mflops	SIP-DG(k) Memory	SIP-DG(k) Mflops
k=1	0.5 Gb	8.723	0.3 Gb	20.040
k=2	0.9 Gb	66.759	2.4 Gb	313.133
k=3	2.6 Gb	309.072	9.3 Gb	2.560.647

Computation 8.3x better, memory 3.5x better: good improvement over DG even if the proposed method is not superconvergent **yet**.

Mesh h	k	Error	Mflops	DOFs	Memory
0.103843	2	3.56e-5	4089984	571392	11.7 Gb
0.207712	3	1.38e-5	309072	115200	2.6 Gb
0.415631	4	1.98e-5	16287	20160	0.5 Gb
0.832917	6	1.24e-5	1265	4032	0.1 Gb

State of HHO for Maxwell equations

HHO on Maxwell is still Work In Progress, but it already works quite well on real-world problems

What we have already

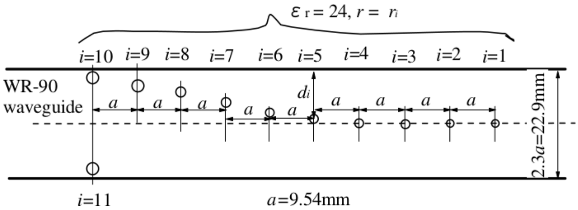
- Dirichlet and Neumann BCs
- Impendance BCs and plane wave sources
- Waveguide sources
- Total field/scattered field formulation

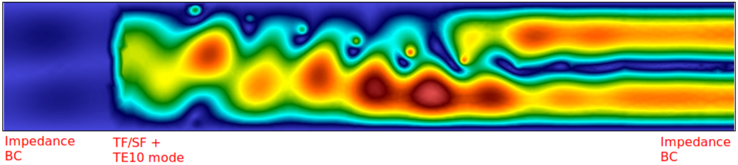
What we do not have yet

- Perfectly Matched Layers (= "numerical materials" used to truncate domain in wave simulations)
- Rigorous mathematical analysis
- Superconvergence

Real world test case: a waveguide mode converter

We study the S_{11} parameter of the depicted mode converted [Kokubo 2011] excited on the left with a TE_{10} mode

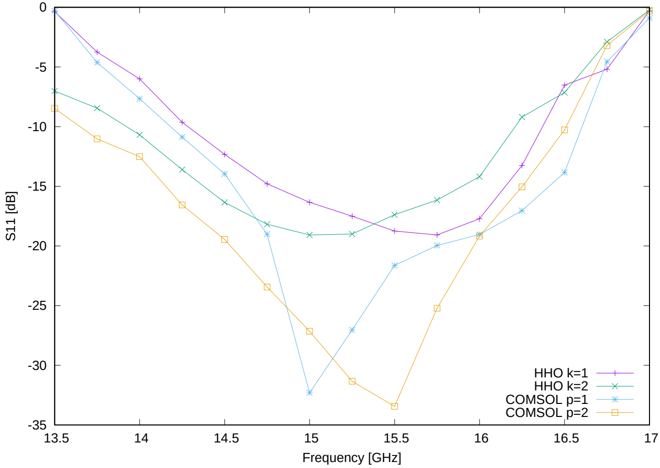




- The simulation showcases all the currently available facilities
- Mesh: 1 layer of triangular prisms generated by GMSH

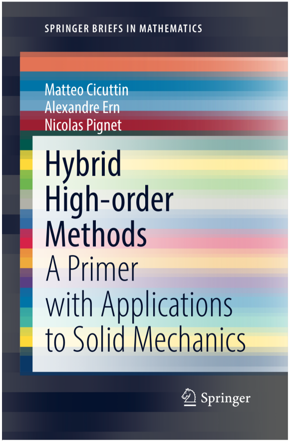
Real world test case: a waveguide mode converter

HHO vs. COMSOL: HHO uses Impedance BC, COMSOL uses PML.



- Decent agreement in the $>$ -20 dB region
- Disagreement in the $<$ -20 dB region because HHO lacks PML

Thank you



¡Gracias por su atención!

An application-oriented book on HHO will be available soon

- Chapters 1-3: Introduction to HHO
- Chapters 4-7: Applications to Solid mechanics
- Chapter 8: Implementation details

Preprint already on HAL & arXiv.

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