A Hybrid High-Order Method For The Indefinite Time-Harmonic Maxwell Problem

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We present a preliminary numerical evaluation of the Hybrid High-Order (HHO) method applied to the indefinite time-harmonic Maxwell problem. HHO is a recently developed member of the family of Discontinuous Sketetal methods, to which belongs also the well-established Hybridizable Discontinuous Galerkin method. HHO provides different valuable assets such as simple construction, support for fully-polyhedral meshes and arbitrary polynomial order, great computational efficiency, physical accuracy and straightforward support for hp-refinement.

***Index Terms*—Propagation, Finite Element Analysis, Hybrid High-Order, time-harmonic Maxwell**

I. Introduction

THE HYBRID HIGH-ORDER method [1], [2] is a recent development in the family of the Discontinuous Skeletal methods which adds to the family of polyhedral discretizations already deployed on computational electromagnetics problems [3], [4], [5]. HHO is already successful in a multitude of fields, including magnetostatics [6]. In this work we present an HHO method for the time-harmonic Maxwell problem. As the time-harmonic Maxwell problem is notoriously hard to solve with iterative methods [7], direct solvers are frequently employed. Direct solvers however require huge amounts of memory, and for this reason efficient, high-order discretization techniques are of utmost importance. By employing skeletal, that is, face-based, unknowns HHO is an excellent candidate for this task.

II. Problem setting

Let Ω be an open, simply connected subset of $\mathbb{R}^3$ (the method is suitable for any spatial dimension, we take $d = 3$ for conciseness). We consider the time-harmonic problem with homogeneous Dirichlet boundary conditions

$$(\mu^{-1}\nabla\times\boldsymbol{e}, \nabla\times\boldsymbol{v})_{L^2(\Omega)} - \omega^2(\epsilon\boldsymbol{e}, \boldsymbol{v})_{L^2(\Omega)} = (\boldsymbol{f}, \boldsymbol{v}), \quad (1)$$

where ω is the angular frequency, μ, ϵ are piecewise constant material parameters, $\boldsymbol{e}, \boldsymbol{v} \in H_0(curl; \Omega)$ are the unknown electric field and the test function respectively; $\boldsymbol{f}$ is the source. A more general setting will be discussed in the full paper.

III. The HHO function spaces

Let $\mathcal{M}(\mathcal{T}, \mathcal{F})$ be a polyhedral mesh with $\#\mathcal{T}$ cells, $\#\mathcal{F}$ faces, maximum element size h, $T \in \mathcal{T}$ a cell and $F \in \mathcal{F}$ a face. We attach to each element T a cell-based vector-valued polynomial $\mathbb{P}_3^k(T)$ and to each one of its n faces $F \in \partial T$ a face-based vector-valued polynomial $\mathbb{P}_2^k(F)$ of degree $k \geqslant 1$. By collecting those polynomials, the element-local space of degrees of freedom is formed and denoted as

$$\mathsf{U}_T^k := \mathbb{P}_3^k(T) \times \left\{ \bigtimes_{F\in\partial T} \mathbb{P}_2^k(F) \right\}.$$

Cell-based polynomials have values in $\mathbb{C}^3$ whereas face-based polynomials have values in $\mathbb{C}^2$ tangent to the face itself. The global discrete problem space is introduced as

$$\mathsf{U}_h^k := \left\{ \bigtimes_{T\in\mathcal{T}} \mathbb{P}_3^k(T) \right\} \times \left\{ \bigtimes_{F\in\mathcal{F}} \mathbb{P}_2^k(F) \right\},$$

where the face-based functions are single-valued. The elements of U_T^k are denoted as the pairs $\underline{\mathsf{u}}_T := (\mathsf{u}_T, \mathsf{u}_{\partial T})$. In turn, u_T and $\mathsf{u}_{\partial T}$ are the cell-based and the collection of face-based polynomials respectively. Similarly, $\underline{\mathsf{u}}_h \in \mathsf{U}_h^k$ and u_h is the cell-based part of U_h^k. Homogeneous Dirichlet boundary conditions are enforced strongly by setting to zero the unknowns associated to the boundary faces:

$$\mathsf{U}_{h,0}^k := \left\{ \underline{\mathsf{u}}_h \in \mathsf{U}_h^k \mid \mathsf{u}_F = 0 \quad \forall F \in \Gamma \right\}.$$

Let also $\gamma_{t,F}(\boldsymbol{u}) := \hat{\boldsymbol{n}} \times (\boldsymbol{u} \times \hat{\boldsymbol{n}})$, with $\hat{\boldsymbol{n}}$ the outward normal.

IV. The HHO operators

The general idea behind skeletal methods is to define an element-local solver which couples to the neighbouring elements via face-based unknowns only. Subsequently, cell-based unknowns are eliminated locally via a Schur complement, obtaining a global transmission problem posed in terms of face unknowns only. In HHO such local solvers are embodied by the *reconstruction operator* [1]. The *curl reconstruction* operator $\mathcal{C} : \mathsf{U}_T^k \to \mathbb{P}_3^k(T)$ is defined as the well-posed problem

$$\begin{aligned}(\mathcal{C}\underline{\mathsf{u}}_T, \boldsymbol{v})_{L^2(T)} &:= (\mathsf{u}_T, \nabla\times\boldsymbol{v})_{L^2(T)} \\ &\quad + \sum_{F\in\partial T} (\mathsf{u}_F, \boldsymbol{v}\times\hat{\boldsymbol{n}})_{L^2(F)}, \quad \forall \boldsymbol{v} \in \mathbb{P}_3^k(T)\end{aligned}$$

The computation of $\mathcal{C}$ requires inverting a mass matrix in each element; this is done just once if a reference element is available. Let π_F^k be the standard face-based L^2-orthogonal projector, let also $\pi_\gamma^k = \pi_F^k \circ \gamma_{t,F}$. The *stabilization* penalizes the difference between the face-based functions and the tangential component of the cell-based function:

$$s_T(\underline{\mathsf{u}}_T, \underline{\mathsf{v}}_T) := \sum_{F\in\partial T} \frac{\kappa^2}{h_F} (\mathsf{u}_F - \pi_\gamma^k(\mathsf{u}_T), \mathsf{v}_F - \pi_\gamma^k(\mathsf{v}_T))_{L^2(F)},$$

where $\kappa^2 = \omega^2\mu\epsilon$ and h_F is the size of the face F.

V. Discrete problem

We use now the curl reconstruction to mimic locally the curl-curl term of (1); we collect this term alongside with the stabilization and the discrete equivalent of the mass term of (1) in the bilinear form

$$\begin{aligned} a_T(\underline{\mathsf{e}}_T, \underline{\mathsf{v}}_T) :=& \mu^{-1}(\mathcal{C}\underline{\mathsf{e}}_T, \mathcal{C}\underline{\mathsf{v}}_T)_{L^2(T)} + s_T(\underline{\mathsf{e}}_T, \underline{\mathsf{v}}_T) \\ & - \omega^2 \epsilon((\mathsf{e}_T, 0), (\mathsf{v}_T, 0))_{L^2(T)} \\ l_T(\underline{\mathsf{v}}_T) :=& (\boldsymbol{f}, (\mathsf{v}_T, 0))_{L^2(T)} \end{aligned}$$

Static condensation is applied locally to eliminate cell-based DOFs, we refer the reader to [8] for the details. The global problem is obtained by a standard finite element assembly as

$$a_h(\underline{\mathsf{e}}_h, \underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}} a_T(\mathsf{L}_T \underline{\mathsf{e}}_h, \mathsf{L}_T \underline{\mathsf{v}}_h),$$

$$l_h(\underline{\mathsf{v}}_h) := \sum_{T \in \mathcal{T}} l_T(\mathsf{L}_T \underline{\mathsf{v}}_h),$$

where L_T is the classical global-to-local element numbering mapping. We finally solve the global discrete problem of finding $\underline{\mathsf{e}}_h \in \mathsf{U}^k_{h,0}$ such that

$$a_h(\underline{\mathsf{e}}_h, \underline{\mathsf{v}}_h) = l_h(\underline{\mathsf{v}}_h) \quad \forall \underline{\mathsf{v}}_T \in \mathsf{U}^k_{h,0}.$$

VI. Conclusions

The described HHO method is implemented in the DiSk++ code (`https://github.com/wareHHOuse/diskpp`) and tested on a resonant cavity problem in the domain $[0,1]^3$. The RHS is chosen to obtain the solution $\boldsymbol{e} = (0, 0, sin(\omega x) sin(\omega y))^T$ with $\omega = \pi$ and $\nu = \epsilon = 1$. The linear system is solved using PARDISO. We observed the expected $O(h^{k+1})$ convergence in L^2-norm and an $O(h^k)$ convergence in energy norm. We compare the convergence rates to a classical Symmetric Interior Penalty Discontinuous Galerkin (SIP-DG) discretization in Figure 1.

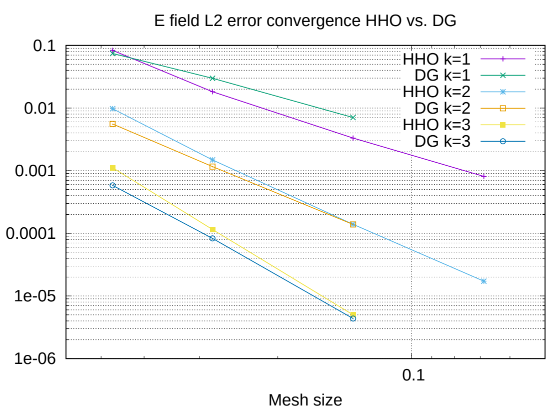

Fig. 1. L^2-norm convergence rates of HHO compared to Symmetric Interior Penalty Discontinuous Galerkin on tetrahedral meshes.

Figure 2 and Table I report the number of operations done by the PARDISO linear solver when deployed on HHO and SIP-DG respectively.

TABLE I
Computational cost comparison between HHO vs. SIP-DG on a tetrahedral mesh of 3072 elements.

	HHO		SIP-DG	
Degree	Memory	Mflops	Memory	Mflops
k=1	0.5 Gb	8.723	0.3 Gb	20.040
k=2	0.9 Gb	66.759	2.4 Gb	313.133
k=3	2.6 Gb	309.072	9.3 Gb	2.560.647

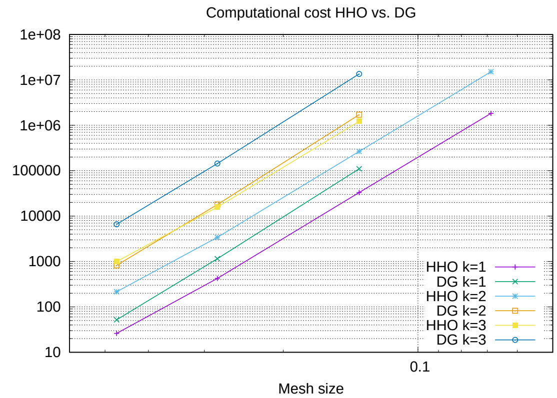

Fig. 2. Number of floating poing operations done by the linear solver. Comparison of HHO vs. SIP-DG on tetrahedral meshes. At high polynomial order, HHO is one order of magnitude cheaper than SIP-DG.

The better performance of HHO is explained by the fact that, by using skeletal unknowns, the number of DOFs grows as $\mathcal{O}(\#\mathcal{F} \cdot k^{d-1})$, compared to $\mathcal{O}(\#\mathcal{T} \cdot k^d)$ in SIP-DG. Moreover, HHO stencil is better suited for the euristics of linear solvers.

We conclude with Table II, in which we analyze the cost of HHO to attain a certain error while varying mesh size and polynomial order.

TABLE II
Computational effort required for HHO to attain roughly the same L^2-norm error at different polynomial orders.

Mesh h	k	Error	Mflops	DOFs	Memory
0.103843	2	3.56e-5	4089984	571392	11.7 Gb
0.207712	3	1.38e-5	309072	115200	2.6 Gb
0.415631	4	1.98e-5	16287	20160	0.5 Gb
0.832917	6	1.24e-5	1265	4032	0.1 Gb

References

[1] D. A. Di Pietro, A. Ern, “A hybrid high-order locking-free method for linear elasticity on general meshes,” *Comp. Meth. Appl. Mech. Eng.* 283, 2015.

[2] M. Cicuttin, A. Ern, N. Pignet, “Hybrid High-Order Methods - A Primer with Applications to Solid Mechanics”, SpringerBriefs in Mathematics, 2021.

[3] B. Cockburn, D. A. Di Pietro and A. Ern, “Bridging the Hybrid High-Order and Hybridizable Discontinuous Galerkin Methods”, *ESAIM Math. Mod. Numer. Anal.*, 50(3), 2016.

[4] N. C. Nguyen, J. Peraire, B. Cockburn, “Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell equations”, *J. Comput. Phys.*, 230, 2011.

[5] M. Cicuttin, R. Specogna, F. Trevisan, “Adaptivity based on the constitutive error for the Maxwell’s eigenvalue problem on polyhedral meshes”, *IEEE Trans. Magn*, 53(6), 2017.

[6] F. Chave, D. A. Di Pietro, S. Lemaire, “A discrete Weber inequality on three-dimensional hybrid spaces with application to the HHO approximation of magnetostatics”, *Preprint, arXiv:2007.03485*, 2020.

[7] M. Cicuttin, L. Codecasa, R. Specogna, F. Trevisan, “A geometric frequency-domain wave propagation formulation for fast convergence of iterative solvers”, *IEEE Trans. Magn*, 53(6), 2017.

[8] M. Cicuttin, D. A. Di Pietro and A. Ern, “Implementation of Discontinuous Skeletal methods on arbitrary-dimensional, polytopal meshes using generic programming”, *J. Comput. Appl. Math.*, 344, 2018.