Complementary discrete geometric h-field formulation for wave propagation problems

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An electromagnetic wave propagation problem can be formulated according to a pair of *complementary* formulations, called $\boldsymbol{e}$-formulation and the $\boldsymbol{h}$-formulation respectively. The two formulations are linked one to each other by the Maxwell's curl equations and, in the continuous setting, they are perfectly equivalent in describing the wave propagation phenomenon. However this is not true in the discrete setting, where the two formulations in general give different solutions. In the past decades complementary formulations were widely studied for static problems and for eddy current problems, where they were exploited as error estimators for adaptive refinement schemes. Moreover, the so-called *bilateral energy bounds* arise for some problems whether theoretically or at least numerically. However, as far as the authors know, little attention was given to complementarity in wave propagation problems. In this work we propose an adaptive refinement scheme using the *constitutive error* as estimator and then we investigate the behaviour in terms of bilateral energy bounds.

***Index Terms*—Discrete Geometric Approach (DGA), Finite Integration Technique (FIT), Wave propagation, Complementarity, Adaptive mesh refinement.**

I. Introduction

An ELECTROMAGNETIC WAVE propagation problem is usually treated in terms of the electric field $\boldsymbol{e}$. In the framework of the Discrete Geometric Approach (DGA) or Finite Integration Technique (FIT) [1], [2], [3], this leads to associate electric voltages to the edges of a primal simplicial grid and magnetomotive forces to the dual barycentric grid edges. However, it is well known that the electromagnetic problem can be also formulated in terms of the magnetic field $\boldsymbol{h}$. In the DGA, the solution of the discrete $\boldsymbol{h}$ problem can be obtained by swapping the role of the simplicial and the barycentric grids, as it will be introduced in this paper.

In the past decades considerable effort has been expended in exploiting complementarity of electromagnetic problems for a number of purposes, in particular for adaptive mesh refinement [4], [5], [6]. Notable results were obtained with static field problems, where bilateral energy bounds are established [7]. Moreover, a wide literature exists about complementarity in eddy current problems. In this last class of problems, however, no energy bounds can be established despite they are frequently observed in pratical problems, see for example [8]. Complementarity in wave propagation problems, seems to have received far less attention. As far as the authors know, it has been investigated only in [5], [9]. In this work we investigated the topic by using the two formulations to build an adaptive refinement scheme based on the approximation lying in the constitutive laws [5]. We then investigated the behaviour in terms of bilateral energy bounds.

In the following sections continuous and discrete wave propagation problem are introduced, both in the $\boldsymbol{e}$-formulation and $\boldsymbol{h}$-formulation. The discrete $\boldsymbol{h}$-formulation is then presented and impedance boundary condition is derived. The adaptive mesh refinement algorithm is then given and numerical results are shown.

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II. Continuous wave propagation problem

From time harmonic Maxwell's equations at angular frequency ω in a bounded domain Ω

$$\nabla \times \boldsymbol{e} = -i\omega \boldsymbol{b}, \qquad \nabla \times \boldsymbol{h} = i\omega \boldsymbol{d},$$

where $\boldsymbol{d}$, $\boldsymbol{e}$, $\boldsymbol{h}$, $\boldsymbol{b}$ are respectively electric displacement, electric, magnetic and magnetic induction fields together with the constitutive relations

$$\boldsymbol{d} = \boldsymbol{\epsilon}\boldsymbol{e}, \qquad \boldsymbol{h} = \boldsymbol{\nu}\boldsymbol{b},$$

where $\boldsymbol{\nu}$ and $\boldsymbol{\epsilon}$ are symmetric positive definite material tensors, the $\boldsymbol{e}$-formulation of electromagnetic wave propagation problem

$$\nabla \times (\boldsymbol{\nu}\nabla \times \boldsymbol{e}) - \omega^2 \boldsymbol{\epsilon}\boldsymbol{e} = \boldsymbol{0}, \tag{1}$$

can be derived [10]. Similarly, the $\boldsymbol{h}$-formulation of the electromagnetic problem becomes

$$\nabla \times (\boldsymbol{\xi}\nabla \times \boldsymbol{h}) - \omega^2 \boldsymbol{\mu}\boldsymbol{h} = \boldsymbol{0}, \tag{2}$$

where $\boldsymbol{\xi} = \boldsymbol{\epsilon}^{-1}$ and $\boldsymbol{\mu} = \boldsymbol{\nu}^{-1}$.

Usual *Perfect Electric Conductor* ($\boldsymbol{n} \times \boldsymbol{e} = \boldsymbol{0}$) and *Perfect Magnetic Conductor* ($\boldsymbol{n} \times \boldsymbol{h} = \boldsymbol{0}$) boundary conditions (BCs) on $\partial\Omega$ with normal $\boldsymbol{n}$ can be applied to the wave propagation problems (1) and (2).

In the $\boldsymbol{e}$-formulation PEC is specified as a Dirichlet BC, while PMC as a Neumann BC. In the $\boldsymbol{h}$-formulation the opposite holds, that is PEC is specified as Neumann condition while PMC as Dirichlet condition [5]. Another BC of interest in the wave propagation problem is the *Impedance boundary condition*, used to constrain the electric and magnetic fields on

a portion of $\partial\Omega$. The numerical treatment of this last condition, in the case of the $\boldsymbol{e}$-formulation, was already presented in [11] leading, by the energetic approach, to the *admittance* matrix $\mathbf{M}_Y$. However the matrix $\mathbf{M}_Y$ is not suitable for the $\boldsymbol{h}$-formulation, so in this paper the *impedance* matrix $\mathbf{M}_Z$ will be derived, again by using the energetic approach.

III. Discrete counterpart of e-formulation

Numerical treatment of (1) requires the discretization of Ω, which is obtained by means of a *primal* tetrahedral grid $\mathcal{G}$ and a barycentric dual grid $\tilde{\mathcal{G}}$. The electromagnetic quantities are associated with these interlocked grids as follows:

- electromotive force U_i to edges $e_i \in \mathcal{G}$;
- magnetic flux Φ_i to faces $f_i \in \mathcal{G}$;
- magnetomotive force F_i to edges $\tilde{e}_i \in \tilde{\mathcal{G}}$;
- electric flux Ψ_i to faces $\tilde{f}_i \in \tilde{\mathcal{G}}$.

Problem (1) is discretized as [1], [2]

$$(\mathbf{C}^T\mathbf{M}_\nu\mathbf{C} - \omega^2\mathbf{M}_\epsilon)\mathbf{U} = \mathbf{0}, \tag{3}$$

where $\mathbf{C}$ is the face-edge incidence matrix, $\mathbf{M}_\nu$ and $\mathbf{M}_\epsilon$ are the constitutive matrices obtained by the energetic approach described in [3] and $\mathbf{U}$ is the array of the unknown voltages along the primal edges. Introducing the impedance boundary conditions, the problem

$$(\mathbf{C}^T\mathbf{M}_\nu\mathbf{C} - \omega^2\mathbf{M}_\epsilon)\mathbf{U} + i\omega\mathbf{M}_Y\mathbf{U} = -2i\omega\mathbf{F}^{b^-}, \tag{4}$$

is obtained, where the term $\mathbf{F}^{b^-}$ represents an excitation applied to a portion of $\partial\Omega$ [2].

IV. Discrete counterpart of h-formulation

The idea behind the complementary wave propagation problem is to exchange the roles of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, by associating:

- electromotive force U_i to edges $\tilde{e}_i \in \tilde{\mathcal{G}}$;
- magnetic flux Φ_i to faces $\tilde{f}_i \in \tilde{\mathcal{G}}$;
- magnetomotive force F_i to edges $e_i \in \mathcal{G}$;
- electric flux Ψ_i to faces $f_i \in \mathcal{G}$.

In this way, complementary discrete Maxwell equations are then written as

$$\mathbf{C}\mathbf{F} = i\omega\mathbf{\Psi}, \tag{5}$$

$$\mathbf{C}^T\mathbf{U} = -i\omega\mathbf{\Phi}, \tag{6}$$

$$\mathbf{U} = \mathbf{M}_\xi\mathbf{\Psi}, \tag{7}$$

$$\mathbf{\Phi} = \mathbf{M}_\mu\mathbf{F}. \tag{8}$$

Solving (5) for $\mathbf{\Psi}$ and substituting it in (7) and then in (6), the complementary wave propagation problem results to be

$$(\mathbf{C}^T\mathbf{M}_\xi\mathbf{C} - \omega^2\mathbf{M}_\mu)\mathbf{F} = \mathbf{0}, \tag{9}$$

where $\mathbf{M}_\xi$ and $\mathbf{M}_\mu$ are obtained by the energetic approach and represent the counterparts of $\mathbf{M}_\nu$ and $\mathbf{M}_\epsilon$, while $\mathbf{F}$ is the magnetomotive force along primal edges. Impedance boundary condition and plane wave excitation can be introduced by adding two terms to (9), obtaining

$$(\mathbf{C}^T\mathbf{M}_\xi\mathbf{C} - \omega^2\mathbf{M}_\mu)\mathbf{F} + i\omega\mathbf{M}_Z\mathbf{F} = 2i\omega\mathbf{U}^{b^-}, \tag{10}$$

where $\mathbf{U}^{b^-}$ is the excitation applied on a portion of $\partial\Omega$. Each nonzero entry of $\mathbf{U}^{b^-}$ corresponds to an edge of $\partial\Omega$. These entries are the voltages due to excitation along each edge of the portion of $\partial\Omega$ where excitation is applied.

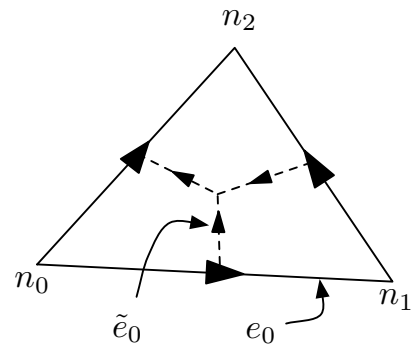

Fig. 1. Orientation of the edges on a boundary $\partial\Omega$. When $\boldsymbol{e}$ is associated to primal edges and $\boldsymbol{h}$ to the dual edges, $e_i^b \times \tilde{e}_i^b$ and $\boldsymbol{e} \times \boldsymbol{h}$ have the same orientation. Swapping the grids, $e_i^b \times \tilde{e}_i^b$ has the orientation of $\boldsymbol{h} \times \boldsymbol{e}$ = -($\boldsymbol{e} \times \boldsymbol{h}$)

A. The impedance boundary condition

At a boundary $\partial\Omega$ where an impedance boundary condition is desired, the equation

$$Z(r)((\boldsymbol{n} \times \boldsymbol{h}) \times \boldsymbol{n}) = \boldsymbol{n} \times \boldsymbol{e} \tag{11}$$

must hold. Moreover, for the properties of the boundary element basis functions $\boldsymbol{v}_i^{e^b}(\boldsymbol{r})$ [11], the equation

$$(\boldsymbol{n} \times \boldsymbol{h}) \times \boldsymbol{n} = \sum_{i=0}^{\mathcal{E}} \boldsymbol{v}_i^{e^b}(\boldsymbol{r}) F_i^b, \tag{12}$$

holds.

B. Constitutive matrix $\mathbf{M}_Z$

The constitutive impedance matrix $\mathbf{M}_Z$ is obtained by the energetic approach [3], [12] in a similar way as done for $\mathbf{M}_Y$ in [11]. Let $\boldsymbol{e'}$ and $\boldsymbol{h}$ be two indipendent fields. We compute the flux of the vector $\boldsymbol{e'} \times \boldsymbol{h}$ across the surface $\partial\Omega$:

$$\begin{aligned}
\int_{\partial\Omega} \boldsymbol{e'} \times \boldsymbol{h}^* \cdot \boldsymbol{n} ds &= \int_{\partial\Omega} \boldsymbol{n} \times \boldsymbol{e'} \cdot \boldsymbol{h}^* ds = \\
&= \int_{\partial\Omega} (\boldsymbol{n} \times \boldsymbol{e'}) \cdot (\boldsymbol{n} \times \boldsymbol{h} \times \boldsymbol{n})^* ds = \\
&= \int_{\partial\Omega} (\boldsymbol{n} \times \boldsymbol{e'}) \cdot (\sum_{i=0}^{\mathcal{E}} \boldsymbol{v}_i^{e^b}(\boldsymbol{r}) F_i^b)^* ds = \\
&= \sum_{i=0}^{\mathcal{E}} F_i^{b^*} \int_{\partial\Omega} (\boldsymbol{n} \times \boldsymbol{e'}) \cdot \boldsymbol{v}_i^{e^b}(\boldsymbol{r}) ds = \\
&= \sum_{i=0}^{\mathcal{E}} F_i^{b^*} \int_{\partial\Omega} (\boldsymbol{v}_i^{e^b}(\boldsymbol{r}) \times \boldsymbol{n}) \cdot \boldsymbol{e'} ds = \\
&= -\sum_{i=0}^{\mathcal{E}} F_i^{b^*} \int_{\partial\Omega} (\boldsymbol{n} \times \boldsymbol{v}_i^{e^b}(\boldsymbol{r})) \cdot \boldsymbol{e'} ds = \\
&= -\sum_{i=0}^{\mathcal{E}} F_i^{b^*} \sum_{j=0}^{\mathcal{F}} \int_{f_j^b} (\boldsymbol{n} \times \boldsymbol{v}_i^{e^b}(\boldsymbol{r})) \cdot \boldsymbol{e'} ds = \\
&= -\sum_{i=0}^{\mathcal{E}} F_i^{b^*} U_i^b = -\boldsymbol{F}^{b'^H} \boldsymbol{U}^{\boldsymbol{b'}}.
\end{aligned} \tag{13}$$

Because of (11) and (12)

$$\int_{\partial\Omega} \boldsymbol{e'} \times \boldsymbol{h}^* \cdot \boldsymbol{n} ds = \tag{14}$$

$$= \int_{\partial\Omega} \left(Z(\boldsymbol{r}) \sum_{i=0}^{\mathcal{E}} \boldsymbol{v}_i^{e^b}(\boldsymbol{r}) F_i^b \right) \left(\sum_{j=0}^{\mathcal{E}} \boldsymbol{v}_j^{e^b}(\boldsymbol{r}) F_j^b \right)^* = \tag{15}$$

$$= \boldsymbol{F}^{b^H} (\mathbf{M}_{\boldsymbol{Z}} \boldsymbol{F}^b) \tag{16}$$

so $\boldsymbol{U}^b = -\mathbf{M}_{\boldsymbol{Z}} \boldsymbol{F}^b$ holds. The entries of the impedance matrix are then calculated as

$$(\mathbf{M}_Z)_{ij} = \int_{\partial\Omega} Z(\boldsymbol{r}) (\boldsymbol{v}_i^{e^b}(\boldsymbol{r}) \cdot \boldsymbol{v}_j^{e^b}(\boldsymbol{r})) ds. \tag{17}$$

V. Adaptive mesh refinement

It is a known fact that in the discrete domain constitutive laws are approximated and thus $\boldsymbol{b}$ is not equal to $\boldsymbol{\mu h}$, as well as $\boldsymbol{d}$ is not equal to $\boldsymbol{\epsilon e}$. For this reason, as already noted by Bossavit, "*this inconsistency in the constitutive laws can be used as an error estimator*" [5]. Thus we propose an adaptive mesh refinement scheme based on the comparison of the electromagnetic energies calculated from the $\boldsymbol{e}$-formulation and $\boldsymbol{h}$-formulation. The main idea behind the scheme is to refine the mesh in the subregions of Ω where the relative error between calculated energies is maximal (Fig. 4), since "*subdivision of the guilty elements and their neighbors, cannot fail to improve the result*" [9, pp. 336-337]. The entire idea can be summarized in the following iterative procedure:

1) solve problems (4) and (10),
2) interpolate fields in the mesh volumes v_i with piecewise constant basis functions [3], obtaining:
   - primal fields $\boldsymbol{e}_{p,i}$, $\boldsymbol{h}_{p,i}$,
   - dual fields $\boldsymbol{e}_{d,i}$ and $\boldsymbol{h}_{d,i}$.
3) for each v_i, let:
   - $\Delta\boldsymbol{e}_i = \boldsymbol{e}_{p,i} - \boldsymbol{e}_{d,i}$,
   - $\Delta\boldsymbol{h}_i = \boldsymbol{h}_{p,i} - \boldsymbol{h}_{d,i}$,

   then compute

$$\Delta w_i = \delta \int_{v_i} \Delta\boldsymbol{e}_i \cdot \epsilon \Delta\boldsymbol{e}_i \, dv + \tag{18}$$

$$(1-\delta) \int_{v_i} \Delta\boldsymbol{h}_i \cdot \mu \Delta\boldsymbol{h}_i \, dv. \tag{19}$$

   The quantity Δw_i represents the absolute energy error between the two formulations in the ith element, while δ is a coefficient in range $[0, \ldots, 1]$.
4) let $\mathcal{T}$ be the set of the tetrahedra in which Ω is discretized:
   - compute

$$w_{p,i} = \delta \int_{v_i} \boldsymbol{e}_{p,i} \cdot \epsilon \boldsymbol{e}_{p,i} \, dv + (1-\delta) \int_{v_i} \boldsymbol{h}_{p,i} \cdot \mu \boldsymbol{h}_{p,i} \, dv$$

   - compute the relative error $\eta(t) = \Delta w_i / w_{p,i}$ for each $v_i \in \mathcal{T}$.
5) assign the tetrahedra of Ω to two sets $\mathcal{T}_h$ and $\mathcal{T}_l$, where the first set contains the $k \cdot 100\%$ of the tetrahedra and maximizes the error, while the second set contains the other tetrahedra. Otherwise stated:
   let $k \in [0, 1]$ and $\eta(\mathcal{X}) = \sum_{x \in \mathcal{X}} \eta(x)$:
   a) make a set $\mathcal{T}_h \subset \mathcal{T}$ such that $card(\mathcal{T}_h) = k \cdot card(\mathcal{T})$ and $\eta(\mathcal{T}_h)$ is maximized,
   b) make a set $\mathcal{T}_l = \mathcal{T} \setminus \mathcal{T}_h$ that contains the remaining tetrahedra,
6) for each tetrahedron $v_i \in \mathcal{T}_h$, divide its radius by r_h,
7) for each tetrahedron $v_i \in \mathcal{T}_l$, divide its radius by r_l.

Good results were obtained by setting $k = 0.1$, $r_h = 3$ and $r_l = 1.2$. The *error weighting coefficient* δ is used to privilege the magnetic energy error ($\delta = 1$) or the electric energy error ($\delta = 0$) in the refinement process. In the first case the refinement captures the rapid variations of $\boldsymbol{e}$, while in the second case the rapid variations of $\boldsymbol{h}$. This could be useful in some cases, for example in presence of strong standing waves (Fig. 2).

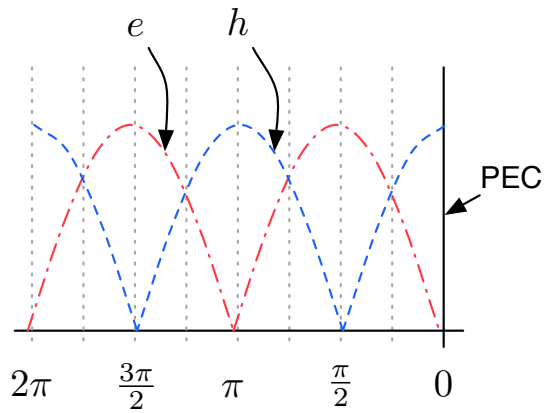

Fig. 2. Standing wave that forms when a plane wave hits a PEC wall. If magnetic field is the quantity of interest, refinement should be done around $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, where the field variations are higher. Electric field, however, would require maximal refinement around 0, π and 2π. Thus, setting $\delta = 0.5$, would lead to a rather uniform refinement while setting $\delta = 0$ or $\delta = 1$ will favour magnetic field or electric field respectively.

VI. Numerical results

We investigated numerically, for a number of wave propagation problems, the convergence behaviour of the two formulations by calculating some energetic quantities at each refinement step, specifically the electric energy [12]

$$w_e = \frac{1}{4} \int_\Omega \epsilon |\boldsymbol{e}|^2 dv = \frac{1}{4} \mathbf{U}^* (\mathbf{M}_\epsilon \mathbf{U}) \tag{20}$$

and the magnetic energy [12]

$$w_m = \frac{1}{4} \int_\Omega \mu |\boldsymbol{h}|^2 dv = \frac{1}{4} \mathbf{F}^* (\mathbf{M}_\mu \mathbf{F}). \tag{21}$$

As an example (Fig. 3), the electric energy of a plane wave travelling in a box of $1m \times 1m \times 1m$ and with an interface with $\Gamma = 0.25$ at the end was calculated. At the second step the refinement procedure produced a mesh of about 8000 elements in the cases $\delta = 0$ and $\delta = 1$, while it produced a mesh of about 12000 elements with $\delta = 0.5$ (uniform refinement). In the case where $\delta = 1$ we observed almost the same accuracy of the case $\delta = 0.5$, despite a 33% reduction of the number of

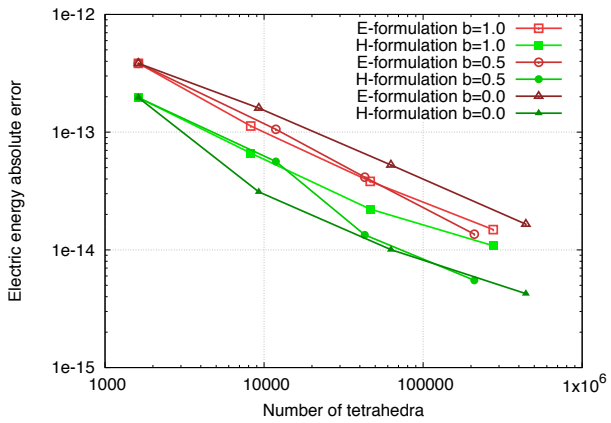

Fig. 3. Comparison of electric energy convergence of the two formulations using different error weighting coefficients: number of tetrahedra vs. absolute energy error. The expected electric energy value is $w_e = 2.35 \cdot 10^{-12}$ J.

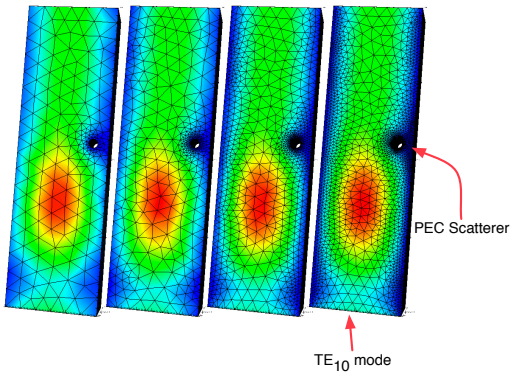

Fig. 4. Four steps of adaptive mesh refinement on a section of rectangular waveguide excited with TE_{10} mode and a scatterer inside. Parameters were $r_h = 3.0, r_l = 1.2, \delta = 1.0$. The adaptive scheme correctly refines the mesh near boundaries, where the variation of the field is higher.

elements (Fig. 3). At the third step the procedure produced a mesh of about 44000 elements in the cases where $\delta = 1$ and $\delta = 0.5$, while it produced a mesh of about 62000 elements in the case $\delta = 0$; with equal number of elements the refinement based on electric energy error is slightly more precise than uniform refinement. Such test problem was chosen to have analytical expressions for the energetic quantities, however the same behaviour was observed in more complex problems, as for example in waveguides with scattering objects inside (Fig. 4), thus suggesting some effectiveness of the proposed technique. Energetic quantities across the domain boundaries were also calculated but, despite the calculation converges to the correct value, inconclusive results were obtained (Fig. 5).

VII. Conclusions

Discrete complementary formulation of wave propagation problem was presented and an adaptive mesh refinement scheme was devised. Bilateral convergence of energetic quantities was also investigated. In the various problems we solved no bilateral energy bounds were observed, as instead happens in static problems or in eddy current problems [8].

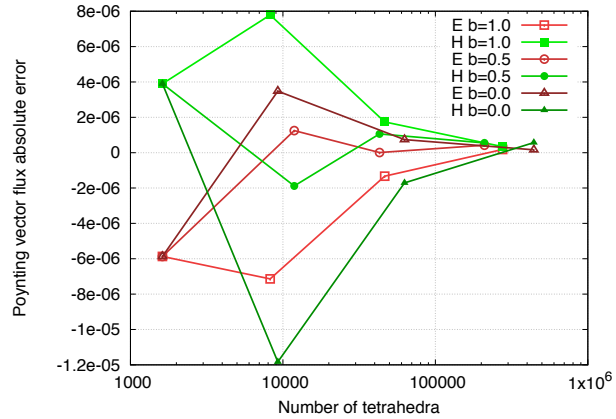

Fig. 5. Comparison of Poynting vector flux convergence of the two formulations using different error weighting coefficients: number of tetrahedra vs. absolute flux error. The expected power flowing across $\partial\Omega$ is $1.244 \cdot 10^{-3}$ W.

Acknowledgements

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